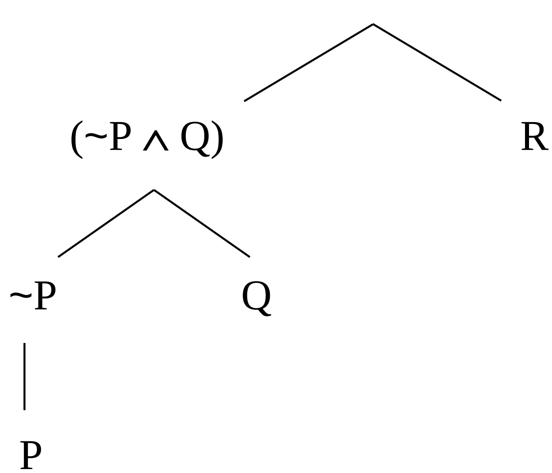
3.10. Scope

1. Scope. Let us return once more to the quote on Stoic logic that began the last section. For while it led us to an appreciation of the main connective, in fact the quote argued its case in somewhat different terms.

> Stoics define ‘contradictories’ as pair of sentences where one exceeds the other by a negative – e.g, “It is day,” “It is not day”. But consider this pair: “It is day and night,” “It is day and not night”; the second exceeds the first by a negative, but they’re not contradictories. But they say: they will be contradictories if the negative is prefixed to the proposition, for then it will have “scope” over [*kurieuei*, governs over] the whole proposition – whereas in “It is day and not night” the negative does not have sufficient scope to negate the whole proposition. (Sextus Empiricus, quoted in Mates 1961: 97)

Here the point is stated in terms of “**scope**”. Like “main connective,” this is a concept easily understood in terms of sentence construction: the **scope** of a connective is the sentence(s) which that connective is attached to in the construction process. So in the following sentence the scope of the tilde is “P”; the scope of the wedge is “~P” and “Q”; and the scope of the vel is “(~P ∧ Q)” and “R”.[1] (A construction tree lists the scope of each connective directly below that connective’s first appearance in the construction process.)

(1) ((~P ∧ Q) ∨ R)



[1] Since the wedge and vel come between two sentences, a wedge or vel has a split, two-part scope – following the usage of, e.g., Kleene 1967/2002: 8.

Facts about the main connective of a sentence can then be restated in terms of scope. For instance, that last sentence is a disjunction because the vel is the connective with the largest scope.

$$(1)\ ((\sim P \wedge Q) \vee R)$$

Indeed, we can provide a new definition of "main connective" in terms of scope.

The **main connective** is the connective with the **widest** (largest) **scope**.

We can likewise rephrase the point from the last section. Intuitively, we judged these two sentences to make very different claims despite their deceptive similarity.

(2) We won't have *both* cake *and* champagne. (2F) $\sim(P \wedge Q)$
(3) We won't have cake, but we'll have champagne. (3F) $(\sim P \wedge Q)$

We now say: sentence (2F) is a negation precisely because **in (2F) the tilde takes wide scope** (compared to the wedge's more **narrow scope**).[2] And (3F) is a conjunction because **in (3F) the wedge takes wide scope** (compared to the tilde's narrow scope).

2. Distribution. When we earlier conjoined or disjoined three or more sentences – forming, for instance, a 'triple-barreled' conjunction or disjunction – the marvelous **associativity** of conjunction and disjunction allowed us to be casual about how the parts of the sentence were grouped. But when mixing different types of connective in the same sentence, we generally cannot be so casual about how the parts get strung together.

The difference between Sentences (2F) and (3F) illustrates that already: we must exercise caution when the connectives in the sentence are a wedge and tilde, or a vel and tilde, and not confuse similar-looking sentences.

[2] We say that a connective's scope is "wide" or "narrow," meaning: compared to the scope of another connective which it's competing with for sentence real estate.

We turn now to equally important differences arising when wedges and vels come together in the same sentence. The following argument, for instance, seems perfectly valid.

VALID	**VALID**
(4) I'll bring either ice cream or cake, **and** <u>I'll bring champagne</u>.	(4F) $((P \vee Q) \wedge \underline{R})$
∴ <u>I'll bring champagne</u>.	∴ $\underline{R}$

This argument is valid for a familiar reason: Sentence (4) is a **conjunction**, and a conjunction validly entails both its left and right parts. Since the conclusion is the right part of Sentence (4), the argument is valid in both English and Formalese.

But that same conclusion doesn't follow validly from Sentence (5) – despite its deceptive similarity to Sentence (4).

INVALID	**INVALID**
(5) **Either** I'll bring ice cream, **or** I'll bring cake and champagne.	(5F) $(P \vee (Q \wedge R))$
∴ I'll bring champagne.	∴ R

Sentence (5) is a **disjunction**, and so doesn't automatically entail its right part. If I assure you of Sentence (5) and show up with ice cream, I've kept my promise. Such a champagne-deprived situation serves as a validity counterexample for this second argument.

Though Sentences (4F) and (5F) differ only in the placement of inner parentheses – hence in terms of how the parts of the sentence are grouped – that makes a world of difference to validity. Clearly **we can't be casual about how the parts are grouped, when mixing wedges and vels** in the same sentence.

Now there *is* a disjunction intuitively equivalent in meaning to Sentence (4): not Sentence (5), but instead Sentence (6), below.

(4) I'll bring either ice cream or cake, **and** I'll bring champagne.	(4F) $((P \vee Q) \wedge R)$
(6) I'll bring **either** ice cream and champagne **or** cake and champagne.	(6F) $((P \wedge R) \vee (Q \wedge R))$

As a promise, Sentence (4) commits me to two things: (i) bringing ice cream or cake, and (ii) bringing champagne. But since there are two ways[3] of fulfilling part (i) – by bringing ice cream, or by bringing cake – keeping promise (4) involves my either bringing ice cream and champagne, or bringing cake and champagne. And that's just what Sentence (6) asserts.

Likewise while Sentence (5) is a disjunction, there *is* a conjunction equivalent in meaning to (5): Sentence (7).

(5) **Either** I'll bring ice cream, **or** I'll bring cake and champagne.	(5F) $((P \vee (Q \wedge R))$
(7) I'll bring either ice cream or cake, **and** I'll bring either ice cream or champagne.	(7F) $((P \vee Q) \wedge (P \vee R))$

(However, in comparing the meaning of 5 and 7 we are pushing the limits of how much complexity we can mentally juggle. Indeed, in this sort of case – where English intuitions about meaning get swamped by complexity – we may find it more useful to let the formal method tell us what the English sentences mean.)[4]

The equivalence of Sentences (4) and (6), and the equivalence of Sentences (5) and (7), are two illustrations of **distribution**. Conjunction **distributes over** disjunction in moving from (4) to (6); and disjunction **distributes over** conjunction in moving from (5) to (7).

[3] Not counting the third, overlap case where I bring *both* ice cream *and* cake (as well as champagne).
[4] In that case formal method will help us understand our own language – rather than waiting on a pre-existing understanding of the English sentences.

To understand distribution *formally,* first compare (4F) with (6F). Beginning with (4F), we take the wedge, and right part following it – “$\wedge$ R” – and attach this to each part of the disjunction “(P $\vee$ Q)”.

$$\text{(4F) } ((P \vee Q) \underline{\wedge R})$$

$$((P \underline{\wedge R}) \vee (Q \underline{\wedge R}) \ \underline{\wedge R})$$

$$\text{(6F) } ((P \underline{\wedge R}) \vee (Q \underline{\wedge R}))$$

Likewise, beginning with Sentence (5) we push “P $\vee$” onto each half of the conjunction “(Q $\wedge$ R)” – yielding (7F).

$$\text{(5F) } ((\underline{P \vee} (Q \wedge R))$$
$$\text{(7F) } ((\underline{P \vee} \ Q) \wedge \underline{(P \vee} R))$$

The equivalences brought by distribution allow us to turn a disjunction (with conjunction as part) into a conjunction (with disjunction as part) – and vice versa.

Distribution

$\mathbf{((P \vee Q) \underline{\wedge R})}$ **is equivalent to** $\mathbf{((P \underline{\wedge R}) \vee (Q \underline{\wedge R}))}$

$\mathbf{((P \wedge Q) \underline{\vee R})}$ **is equivalent to** $\mathbf{((P \underline{\vee R}) \wedge (Q \underline{\vee R}))}$

To put that same point in term of scope: “((P $\vee$ Q) $\wedge$ R),” where the wedge has wider scope, is equivalent to “((P $\wedge$ Q) $\vee$ R),” where the vel has wider scope. Distribution allows us to **reverse the hierarchy of scope** when faced with competing wedges and vels.

3. English Exceptions: Relative Clauses and "Without," Revisited. In closing we use our understanding of scope to shed further light on some puzzling sentences of English.

"Without". Recall that a "without" sentence is translated as a conjunction with negated right half (though this right half needs a bit of reconstruction to tease out the subject matter sentence involved).

R: Rex passed Chemistry
S: Rex studied

(8) Rex passed Chemistry without studying. (8F) $(R \wedge {\sim}S)$

Now Sentence (9) is word-for-word the same as (8), except with "didn't" added to its left side.

(9) Rex **didn't** pass Chemistry without studying.

However, formal translation of (9) is not as simple as it might seem. It looks as though "didn't" applies *only on the left part* of Sentence (8).

☠ **Correct Translation??** ☠

(9) Rex **didn't** pass Chemistry without studying. (9F?) $({\sim}R \wedge {\sim}S)$

Reading the negation that way treats (9) as a conjunction. But is that right?

We know a conjunction validly entails each of its parts. Sentence (8), for instance, is treated as a conjunction; and (8) does indeed validly entail its left half.

VALID	**VALID**
(8) <u>Rex passed Chemistry</u> without studying.	(8F) $(\underline{P} \wedge {\sim}Q)$
∴ <u>Rex passed Chemistry</u>.	∴ $\underline{P}$

Yet we don't find this entailment with Sentence (9). In a situation where Rex passed Chemistry only after intensive studying, Sentence (9) is true; but it's false there that "Rex didn't pass Chemistry". That serves as a validity counterexample for this argument.

INVALID!

(9) Rex **didn't** pass Chemistry without studying.

∴ Rex didn't pass Chemistry.

By contrast, the formal sentence "$(\sim R \wedge \sim S)$" certainly *does* entail its left half, "$\sim R$" – suggesting that Sentence (9) should not be translated as "$(\sim R \wedge \sim S)$".

In fact, English speakers interpret (9) instead as the **denial** (negation) **of (8)**. If someone claims Rex didn't earn his passing grade, but just lucked into it without studying, we deny that scurrilous accusation by uttering Sentence (9): "*Rex didn't pass Chemistry without studying* – on the contrary, he studied like crazy!"

That means Sentence (9) should be translated as follows.

(8) Rex passed Chemistry without studying.	(8F)	$(R \wedge \sim S)$
(9) Rex **didn't** pass Chemistry without studying.	(9F)	$\sim(R \wedge \sim S)$

In effect, a "without" sentence of English, such as Sentence (8), acts like a sealed conjunction whose borders can't be breached by a tilde. So while it *appears* in English that "not" applies just to the left half of Sentence (9), in fact "not" is locked outside, denying everything that follows. "Not" thus takes **wide scope** (compared to the wedge) – making (9) a negation.

Here the order of parts in English is *not* a reliable clue to the scope of the competing bits of form.

Relative Clauses. We noted that in formal translation we could treat a sentence with a *relative clause* as a kind of conjunction in disguise (provided we reconstruct the right half a bit).

P: Jack is a dog
Q: Jack eats flies

(10) Jack is a dog who eats flies. (10F) $(P \wedge Q)$

Now Sentence (11) is word-for-word the same as (10), except with "not" added to its left side.

(11) Jack **isn't** a dog who eats flies.

And since (11) is still a sentence with a relative clause, we might expect it to translate as a conjunction (though now one with a negated left half). That is: English suggests that the wedge takes wider scope in formal translation, and the **tilde** takes **narrow** scope.

☠ Correct Translation?? ☠

(11) Jack **isn't** a dog who eats flies. $(\sim P \wedge Q)$

But here again appearances deceive; for in fact (11) should *not* be treated as a conjunction with negated left half. As evidence we contrast (11) with (12) – a sentence which definitely *is* a conjunction with negated left half.

(12) Jack isn't a dog, but Jack eats flies. (10F) $(\sim P \wedge Q)$

Once again, a conjunction such as (12) entails its right half.

VALID	**VALID**
(12) Jack isn't a dog, but <u>Jack eats flies</u>.	(12F) $(\sim P \wedge Q)$
∴ <u>Jack eats flies</u>.	∴ <u>Q</u>

But Sentence (11) doesn't entail that Jack eats flies. In a situation where Jack is a cat who doesn't eat flies, Sentence (11) is true but the conclusion is false – yielding a validity counterexample for this argument.

INVALID

(11) Jack isn't a dog who eats flies.

∴ <u>Jack eats flies</u>.

If Sentence (11) *were* a conjunction with negated left half, just like (12), then (11) ought to entail "Jack eats flies" just as (12) does. Since that's not the case, we conclude that (11) doesn't have the same logical form as Sentence (12).

Instead, (11) is the denial – the **negation** – of Sentence (10). (11) is thus translated as (11F) – giving the wedge narrow scope, and the **tilde wide scope**.

(10) Jack is a dog who eats flies.	(10F) $(P \wedge Q)$
(11) Jack **isn't** a dog who eats flies.	(11F) $\sim(P \wedge Q)$

Again this agrees with the understanding English speakers have of Sentence (11): if someone claims that Jack is a fly-eating dog, I can deny that by uttering (11).

"Jack is a dog who eats flies."
"That's a dirty lie – *Jack isn't a dog who eats flies*!"

An English sentence with a relative clause acts like a sealed conjunction, whose borders can't be breached by a tilde. So while it *appears* in English that "not" applies just to the left half of Sentence (11), in fact "not" is locked outside, denying everything that follows. "Not" thus takes **wide scope** (compared to the conjunction) – making (11) a negation.

(11) Jack **isn't** a dog who eats flies. (11F) $\sim(P \wedge Q)$

Here again the order of parts in English is *not* a reliable clue to the scope of the competing bits of form.

None of that is the fault of the formal language, of course. Every formal connective wears its scope on its sleeve (and its parentheses), with no opportunity for deception. This is instead another curiosity of English (further study of which would take us too far afield).[5] The peculiarities of "without" sentences and relative clause-bearing sentences are relevant here mainly as a further application of the concepts of scope and the main connective.

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[5] Briefly: though we spoke above of , e.g., the 'left part' of a "without" sentence, linguistically there's no such thing. For instance, "Rex passed Chemistry" isn't really the 'left part' of "Rex passed Chemistry without studying" – indeed, grammatically it's not a *natural part* of that "without" sentence at all. (In linguistic jargon: it's not a **constituent** of the larger "without" sentence.) Here arises a fundamental structural difference between natural languages like English and artificial languages like Formalese: while the formal language allows marvelously symmetrical sentences such as "$(P \wedge Q)$," English sentences seem to be intrinsically *asymmetrical*.